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NONLINEAR THEORIES FOR THIN SHELLS

By

J. Lyell Sanders, Jr.

Division of Engineering and Applied Physics
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ABSTRACT

Strain-displacement relations for thin shells valid for large displacements are derived. With these as a starting point approximate strain-displacement relations and equilibrium equations are derived by making certain simplifying assumptions. In particular the middle surface strains are assumed small and the rotations are assumed moderately small. The resulting equations are suitable as a starting point for stability investigations or other problems in which the effects of deformation on equilibrium cannot be ignored, but in which the rotations are not too large.

The linearized forms of several of the sets of equations derived herein coincide with small deflection theories in the literature.

INTRODUCTION

One of the important uses to which a large displacement theory of thin shells can be put is the investigation of stability. The many papers in the literature on the stability of shells have dealt almost exclusively with cylinders, spheres, and cones, and the differential equations governing the phenomenon have been derived specifically for these geometrical shapes. It would seem to be desirable to have a unified treatment based on a general theory for an arbitrary middle surface. The practically important cases of the shallow shell and the shell of revolution with symmetric deformations have been adequately treated (references 1 and 2) but the general problem

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presents difficulties not found in the special cases. It is the purpose of the present paper to derive an exact theory for large deflections of a thin shell with an arbitrary middle surface and then by making certain simplifying assumptions to derive from this several theories suitable for application.

An incomplete treatment of the general large deflection theory of thin shells has been given by Novozhilov in reference 3. He derives a theory for small middle surface strains but does not go into detail on further simplifications or discuss approximate equilibrium equations. He indicates that the next step is to assume that the rotations are small. His results for the small strain theory differ from those in the present paper.

In the western literature there have been several papers dealing with the general large deflection theory, in particular the paper by Synge and Chien (reference 4), the series of papers by Chien (references 5 and 6) and the recent paper by Ericksen and Truesdell (reference 7). The intrinsic theory of shells developed by Synge and Chien avoids the use of displacements as unknowns of the problem. The theory of shells is deduced from the three dimensional theory of elasticity and then by means of series expansions in powers of a small thickness parameter approximate theories of thin shells are derived. A large number of problem types is found classified according to the relative orders of magnitude of various quantities. This approach has been criticized by several authors (see references 8, 9 and 10). In the paper of Ericksen and Truesdell there is a unified treatment of shells and curved rods developed as two and one dimensional theories respectively without an attempt to deduce these from the three dimensional theory of elasticity. The constitutive relations are purposely left out of consideration since they are unnecessary for the description of strain and the establishment of equilibrium conditions. The authors are interested only in exact theory and do not discuss the

simplifications of small strains and rotations.

The papers of Synge and Chien and that of Ericksen and Truesdell on the shell problem are unorthodox and difficult to relate to most of the rest of the literature. The authors have aimed at a maximum of generality, perhaps more than necessary for the technological applications of the theory.

In the present paper a large deflection theory for thin shells is developed in which transverse shear and normal strains are neglected. Without further approximations the equations are very complicated, but since large middle surface strains are almost never encountered in engineering applications, an effort is made to derive simpler equations based on the assumption of small strains. Considerably more simplification is gained when the rotations as well as the middle surface strains are assumed to be small. In this way a theory for shells is derived comparable to the von Karman theory for plates. The equations corresponding to other minor simplifications are also presented. The main part of the paper is written in tensor notation but the principal results are reproduced in the ordinary notation in an appendix.

DERIVATION OF EXACT EQUATIONS FOR LARGE DISPLACEMENTS

Geometrical Preliminaries

Let the undeformed middle surface of the shell be given by the equations

$$x^i = x^i(\xi^\alpha) \qquad (i = 1, 2, 3\,;\ \alpha = 1, 2) \tag{1}$$

where the x^i are cartesian coordinates in space and the ξ^α are curvilinear coordinates on the surface. Let the displacements U^i of material points on the middle surface of the shell be resolved into components tangential and normal to the undeformed middle surface as expressed by the following equation

$$U^i = U^\alpha(\xi)x^i_{,\alpha} + W(\xi)N^i \qquad (2)$$

where $x^i_{,\alpha} = \frac{\partial x^i}{\partial \xi^\alpha}$ are tangent vectors to the coordinate curves on the undeformed middle surface and N^i is the unit normal to the undeformed middle surface. In this paper the coordinates ξ^α will be used to label material particles on both the undeformed middle surface and the deformed middle surface.

Some of the important formulas of the theory of surfaces will be used repeatedly and are reproduced here for convenient reference. For the undeformed middle surface the formula for the squared element of arc ds^2 in terms of the first fundamental form $g_{\alpha\beta}$ is

$$ds^2 = x^i_{,\alpha}x^i_{,\beta}\, d\xi^\alpha d\xi^\beta = g_{\alpha\beta}\, d\xi^\alpha d\xi^\beta \qquad (3)$$

the element of area is

$$dA = \sqrt{g}\, d\xi^1 d\xi^2 \qquad (4)$$

where g is the determinant of $g_{\alpha\beta}$; and the equations of Gauss, Weingarten and Codazzi are

$$x^i_{,\alpha\beta} = - b_{\alpha\beta}N^i \qquad (5)$$

$$N^i_{,\alpha} = b^\beta_\alpha x^i_{,\beta} \qquad (6)$$

$$b_{\alpha\beta,\gamma} = b_{\alpha\gamma,\beta} \qquad (7)$$

where a comma denotes covariant differentiation with respect to the metric $g_{\alpha\beta}$ and where the second fundamental form $b_{\alpha\beta}$, as here used, differs in sign from the usual definition.

After the displacement U^i given by equation (2), the material particle originally at x^i will move to the point y^i given by

$$y^i = x^i + U^i$$
$$= x^i + U^\alpha x^i_{,\alpha} + WN^i \tag{8}$$

This is the equation of the deformed middle surface in terms of the parameters ξ^α. Tangent vectors to the coordinate curves on the deformed middle surface are given by

$$y^i_{,\alpha} = \lambda^\gamma_\alpha x^i_{,\gamma} + \mu_\alpha N^i \tag{9}$$

where

$$\lambda_{\alpha\beta} = x^i_{,\alpha} y^i_{,\beta} = g_{\alpha\beta} + U_{\alpha,\beta} + b_{\alpha\beta} W \tag{10}$$

$$\lambda^\gamma_\alpha \equiv \lambda^\gamma_{.\alpha} = g^{\gamma\delta}\lambda_{\delta\alpha} \tag{11}$$

$$\mu_\alpha = y^i_{,\alpha} N^i = W_{,\alpha} - b^\beta_\alpha U_\beta \tag{12}$$

Also define

$$\nu_\alpha = x^i_{,\alpha}\bar{N}^i \tag{13}$$

$$\cos\omega = N^i\bar{N}^i \tag{14}$$

where $\bar{N}^i$ is the unit normal to the deformed middle surface (see equation (73)). Indices on $\lambda_{\alpha\beta}$, μ_α and ν_α will always be raised or lowered with the metric $g_{\alpha\beta}$. Where necessary a bar indicates a quantity defined with respect to the deformed middle surface. The squared element of arc on the deformed middle surface is given by

$$d\bar{s}^2 = y^i_{,\alpha} y^i_{,\beta}\, d\xi^\alpha d\xi^\beta = G_{\alpha\beta}\, d\xi^\alpha d\xi^\beta \tag{15}$$

In terms of $\lambda_{\alpha\beta}$ and μ_α

$$G_{\alpha\beta} = \lambda^\gamma_\alpha \lambda_{\gamma\beta} + \mu_\alpha\mu_\beta \tag{16}$$

The element of area $d\bar{A}$ on the deformed middle surface is given by

$$d\bar{A} = \sqrt{G}\, d\xi^1 d\xi^2 = \sqrt{\frac{G}{g}}\, dA \tag{17}$$

The equations of Gauss, Weingarten and Codazzi are

$$y^i_{/\alpha\beta} = - B_{\alpha\beta} N^i \tag{18}$$

$$N^i_{,\alpha} = B^\beta_\alpha y^i_{,\alpha} \tag{19}$$

$$B_{\alpha\beta/\gamma} = B_{\alpha\gamma/\beta} \tag{20}$$

where $B_{\alpha\beta}$ is the second fundamental form of the deformed middle surface. A slash is used to denote covariant differentiation with respect to the metric $G_{\alpha\beta}$. For scalars and cartesian tensors (as W, y^i or N^i) there is no difference between a slash and a comma. An expression for $B_{\alpha\beta}$ in terms of $\lambda_{\alpha\beta}$, μ_α, ν_α and $\cos\omega$ can be derived as follows

$$\begin{aligned} B_{\alpha\beta} &= y^i_{,\alpha} N^i_{,\beta} = - y^i_{,\alpha\beta} N^i \\ &= -[(\lambda^\gamma_{\alpha,\beta} + b^\gamma_\beta \mu_\alpha) x^i_{,\gamma} + (\mu_{\alpha,\beta} - \lambda^\gamma_\alpha b_{\gamma\beta}) N^i] N^i \\ &= (b^\gamma_\beta \lambda_{\gamma\alpha} - \mu_{\alpha,\beta}) \cos\omega - (\lambda_{\gamma\alpha,\beta} + b_{\beta\gamma}\mu_\alpha)\nu^\gamma \end{aligned} \tag{21}$$

The following identity will prove to be useful

$$y^i_{,\alpha} N^i = \nu^\beta \lambda_{\beta\alpha} + \mu_\alpha \cos\omega = 0 \tag{22}$$

Equilibrium Equations

In the coordinate system of the deformed middle surface the equilibrium equations of the shell are the same as in the linear theory and need not be derived here. They are (see reference 10):

force equilibrium

$$N^{\alpha\beta}_{/\alpha} + B^\beta_\alpha Q^\alpha + \bar{p}^\beta = 0 \tag{23}$$

$$Q^\alpha_{/\alpha} - B_{\alpha\beta} N^{\alpha\beta} + \bar{p} = 0 \tag{24}$$

moment equilibrium

$$M^{\alpha\beta}{}_{/\alpha} - Q^{\beta} = 0 \tag{25}$$

$$\bar{\varepsilon}_{\alpha\beta}(N^{\alpha\beta} + B^{\alpha}_{\gamma}M^{\gamma\beta}) = 0 \tag{26}$$

where $N^{\alpha\beta}$ is the membrane stress resultant, $M^{\alpha\beta}$ is the bending moment resultant, and Q^{α} is the transverse shear stress resultant, all defined with respect to the deformed shell. The quantities $\bar{p}^{\alpha}$ and $\bar{p}$ are applied load intensities per unit of area of the deformed middle surface, $\bar{\varepsilon}_{\alpha\beta}$ is the covariant permutation tensor in the deformed coordinate system.

The above equilibrium equations are exact but, of course, the ten stress quantities entering into them do not furnish a complete description of the state of stress throughout the thickness of the shell. However, in thin shell theory it is always assumed that the state of stress is adequately described in terms of these quantities.

Finite Strains

The strain quantities entering into a thin shell theory are a matter for definition. The literature of the subject shows a wide variety of choices of strain-displacement relations, particularly for the bending strains. Some choices have been shown to be better than others (see reference 11) but at the present time no set of conditions sufficient to render the choice unique has been generally agreed upon. In the present paper the choice has been guided by two considerations, the first of which was the desire to derive a theory which admits a principle of virtual work. This requirement forces a close relation between the equilibrium equations and the strain-displacement relations. The second consideration was simplicity. The resultant cnoice will be shown to furnish an adequate description of the deformation of the shell provided the Kirchoff hypotheses are accepted as adequate descriptions of the displacements.

Let $\bar{S}$ be a simply connected region on the deformed middle surface enclosed by the curve $\bar{C}$. The following identity follows from equations (23) to (26).

$$\int_{\bar{S}} [(N^{\alpha\beta}{}_{/\alpha}+B^{\beta}_{\alpha}Q^{\alpha}+\bar{p}^{\beta})\delta\bar{U}_{\beta} + (Q^{\alpha}{}_{/\alpha}-B_{\alpha\beta}N^{\alpha\beta}+\bar{p})\delta\bar{W}$$

$$+ (M^{\alpha\beta}{}_{/\alpha}-Q^{\beta})\delta\phi_{\beta} + \bar{\varepsilon}_{\alpha\beta}(N^{\alpha\beta}+B^{\alpha}_{\gamma}M^{\gamma\beta})\delta\phi]d\bar{A} = 0 \qquad (27)$$

By application of the divergence theorem for a curved surface (27) may be transformed into the following identity which is the preliminary form of the principle of virtual work and all subsequent derivations will proceed from it.

$$\oint_{\bar{C}} (N^{\alpha\beta}\delta\bar{U}_{\beta}+Q^{\alpha}\delta\bar{W}+M^{\alpha\beta}\delta\phi_{\beta})\bar{n}_{\alpha}\, d\bar{s} + \int_{\bar{S}} (\bar{p}^{\beta}\delta\bar{U}_{\beta}+\bar{p}\delta\bar{W})d\bar{A}$$

$$= \int_{\bar{S}} [N^{\alpha\beta}(\delta\bar{U}_{\beta/\alpha}+B_{\alpha\beta}\delta\bar{W}-\bar{\varepsilon}_{\alpha\beta}\delta\phi) + Q^{\alpha}(\delta\bar{W}_{/\alpha}-B^{\beta}_{\alpha}\delta\bar{U}_{\beta}+\delta\phi_{\alpha})$$

$$+ M^{\alpha\beta}(\delta\phi_{\beta/\alpha}-\bar{\varepsilon}_{\gamma\beta}B^{\gamma}_{\alpha}\delta\phi)]d\bar{A} \qquad (28)$$

where the virtual displacements $\delta\bar{U}_{\alpha}$, $\delta\bar{W}$ and rotations $\delta\phi_{\alpha}$, $\delta\phi$ refer to components in the directions of the tangents and normal to the deformed middle surface. In (28) the terms on the left hand side are interpretable as the external virtual work of edge loads and surface loads respectively. The right hand side of (28) might be interpreted as internal virtual work if the coefficients of $N^{\alpha\beta}$, Q^{α} and $M^{\alpha\beta}$ were identified with strain increments. Such an identification will be postponed until later. First these coefficients will be written in a different form.

By definition

$$\delta U^{i} = \delta\bar{U}^{\alpha}y^{i}_{,\alpha} + \delta\bar{W}\,\bar{N}^{i} = \delta U^{\alpha}x^{i}_{,\alpha} + \delta W\, N^{i} \qquad (29)$$

Now

$$\delta U^i_{/\beta} = (\delta\overline{W}_{/\beta} - B_{\alpha\beta}\delta\overline{U}^\alpha)\overline{N}^i + (\delta\overline{U}^\alpha_{/\beta} + B^\alpha_\beta\delta\overline{W})y^i_{,\alpha} = \delta U^i_{,\beta}$$

$$= (\delta W_{,\beta} - b_{\alpha\beta}\delta U^\alpha)N^i + (\delta U^\alpha_{,\beta} + b^\alpha_\beta\delta W)x^i_{,\alpha} \qquad (30)$$

From which:

$$\delta\overline{U}_{\beta/\alpha} + B_{\alpha\beta}\delta\overline{W} = \lambda_{\gamma\beta}\delta\lambda^\gamma_\alpha + \mu_\beta\delta\mu_\alpha \qquad (31)$$

$$\delta\overline{W}_{/\alpha} - B^\gamma_\alpha\delta\overline{U}_\gamma = \nu_\gamma\delta\lambda^\gamma_\alpha + \cos\omega\ \delta\mu_\alpha \qquad (32)$$

The rotation around the normal $\delta\phi$ is given in terms of displacements by the formula

$$\delta\phi = \frac{1}{2}\bar{\varepsilon}^{\alpha\beta}\delta\overline{U}_{\beta/\alpha} \qquad (33)$$

By the use of (31) and the fact that $B_{\alpha\beta}$ is symmetric (33) becomes

$$\delta\phi = \frac{1}{2}\bar{\varepsilon}^{\alpha\beta}(\lambda_{\gamma\beta}\delta\lambda^\gamma_\alpha + \mu_\beta\delta\mu_\alpha) \qquad (34)$$

Also $$\bar{\varepsilon}_{\alpha\beta}\delta\phi = \frac{1}{2}(\lambda_{\gamma\beta}\delta\lambda^\gamma_\alpha + \mu_\beta\delta\mu_\alpha - \lambda_{\gamma\alpha}\delta\lambda^\gamma_\beta - \mu_\alpha\delta\mu_\beta) \qquad (35)$$

From (31) and (35) the coefficient of $N^{\alpha\beta}$ in (28) is

$$\frac{1}{2}(\lambda_{\gamma\beta}\delta\lambda^\gamma_\alpha + \mu_\beta\delta\mu_\alpha + \lambda_{\gamma\alpha}\delta\lambda^\gamma_\beta + \mu_\alpha\delta\mu_\beta) = \frac{1}{2}\,\delta(\lambda_{\gamma\beta}\lambda^\gamma_\alpha + \mu_\alpha\mu_\beta) = \frac{1}{2}\,\delta G_{\alpha\beta} \qquad (36)$$

The natural definition of the finite membrane strain is thus

$$E_{\alpha\beta} = \frac{1}{2}(G_{\alpha\beta} - g_{\alpha\beta}) \qquad (37)$$

The coefficient of Q^α in (28) is

$$\delta\gamma_\alpha = \delta\overline{W}_{/\alpha} - B^\beta_\alpha\delta\overline{U}_\beta + \delta\phi_\alpha = \nu_\gamma\delta\lambda^\gamma_\alpha + \cos\omega\ \delta\mu_\alpha + \delta\phi_\alpha \qquad (38)$$

Since the intention is to derive a theory in which transverse shear strains are neglected, set $\delta\gamma_\alpha = 0$ which gives

$$\delta\phi_\alpha = -\nu_\gamma\delta\lambda^\gamma_\alpha - \cos\omega\ \delta\mu_\alpha \qquad (39)$$

which serves to relate rotations to displacements. From (9), (13), (14) and (39)

$$\bar{N}^i \delta y^i_{,\alpha} = \nu_\gamma \delta\lambda^\gamma_\alpha + \cos\omega\, \delta\mu_\alpha = -\delta\phi_\alpha \tag{40}$$

or since $\bar{N}^i y^i_{,\alpha} = 0$ it follows that

$$\delta\phi_\alpha = y^i_{,\alpha}\delta\bar{N}^i \tag{41}$$

A finite transverse shear strain γ_α consistent with (38) and the requirement $\gamma_\alpha = 0$ may be defined as follows

$$\begin{aligned}\gamma_\alpha &= \bar{N}^i y^i_{,\alpha} \\ &= \lambda^\gamma_\alpha \nu_\gamma + \mu_\alpha \cos\omega\end{aligned} \tag{42}$$

The coefficient of $M^{\alpha\beta}$ in (28) may be found in terms of $B_{\alpha\beta}$ and $G_{\alpha\beta}$ as follows. From (41)

$$\begin{aligned}\delta\phi_{\beta/\alpha} &= y^i_{/\beta\alpha}\delta\bar{N}^i + y^i_{,\beta}\delta\bar{N}^i_{,\alpha} \\ &= -B_{\alpha\beta}\bar{N}^i\delta\bar{N}^i + y^i_{,\beta}\delta\bar{N}^i_{,\alpha} \\ &= y^i_{,\beta}\delta\bar{N}^i_{,\alpha}\end{aligned} \tag{43}$$

since $\bar{N}^i\delta\bar{N}^i = 0$. Now recall

$$B_{\alpha\beta} = \bar{N}^i_{,\alpha} y^i_{,\beta}$$

so

$$\begin{aligned}\delta B_{\alpha\beta} &= y^i_{,\beta}\delta\bar{N}^i_{,\alpha} + \bar{N}^i_{,\alpha}\delta y^i_{,\beta} \\ &= \delta\phi_{\beta/\alpha} + B^\gamma_\alpha y^i_{,\gamma}\delta y^i_{,\beta}\end{aligned} \tag{44}$$

Which gives

$$\delta\phi_{\beta/\alpha} = \delta B_{\alpha\beta} - B^\gamma_\alpha y^i_{,\gamma}\delta y^i_{,\beta} \tag{45}$$

From (9) and (35)

$$\bar{\varepsilon}_{\alpha\beta}\delta\phi = \frac{1}{2}(y^i_{,\beta}\delta y^i_{,\alpha} - y^i_{,\alpha}\delta y^i_{,\beta}) \tag{46}$$

From (45) and (46) the coefficient of $M^{\alpha\beta}$ in (28) reduces to

$$\delta B_{\alpha\beta} - \frac{1}{2} B^{\gamma}_{\alpha}\delta(y^{i}_{,\beta}y^{i}_{,\gamma})$$

$$= \delta B_{\alpha\beta} - \frac{1}{2} B^{\gamma}_{\alpha}\delta G_{\beta\gamma}$$

$$= \delta B_{\alpha\beta} - B^{\gamma}_{\alpha}\delta E_{\beta\gamma} \tag{47}$$

Using the foregoing results the right hand side of (28) may be written

$$\int_{\bar{S}} [N^{\alpha\beta}\delta E_{\alpha\beta} + M^{\alpha\beta}(\delta B_{\alpha\beta} - B^{\gamma}_{\alpha}\delta E_{\beta\gamma})]d\bar{A} \tag{48}$$

There is obviously some difficulty in defining a finite bending strain tensor because the coefficient of $M^{\alpha\beta}$ in this expression is not the exact variation of anything. However, a way to proceed suggests itself if (48) is rewritten in the following form

$$\int_{\bar{S}} [(N^{\alpha\beta} - B^{\beta}_{\gamma}M^{\gamma\alpha})\delta E_{\alpha\beta} + M^{\alpha\beta}\delta B_{\alpha\beta}]d\bar{A} \tag{49}$$

Define a finite bending strain tensor by

$$K_{\alpha\beta} = B_{\alpha\beta} - b_{\alpha\beta} \tag{50}$$

Define a modified membrane stress tensor by

$$\tilde{N}^{\alpha\beta} = N^{\alpha\beta} - B^{\beta}_{\gamma}M^{\gamma\alpha} \tag{51}$$

Since $B_{\alpha\beta}$ is symmetric there will be no loss in generality by defining a modified bending moment tensor by

$$\tilde{M}^{\alpha\beta} = \frac{1}{2}(M^{\alpha\beta} + M^{\beta\alpha}) \tag{52}$$

Note that the third moment equilibrium equation (equation (26)) is equivalent to the statement that $\tilde{N}^{\alpha\beta}$ is symmetric. In terms of the newly defined quantities (49) becomes

$$\int_{\bar{S}} (\tilde{N}^{\alpha\beta}\delta E_{\alpha\beta} + \tilde{M}^{\alpha\beta}\delta K_{\alpha\beta})d\bar{A} \tag{53}$$

The details will not be shown here but this expression for the internal virtual work may be derived from the three dimensional theory by integration through the thickness of the shell and <u>without approximation</u> provided the displacements are restricted by the Kirchoff hypotheses. The appearance of the modified tensors $\tilde{N}^{\alpha\beta}$ and $\tilde{M}^{\alpha\beta}$ rather than $N^{\alpha\beta}$ and $M^{\alpha\beta}$ may seem somewhat strange but then there is no reason why these quantities should not be adopted as the stress quantities entering into the theory rather than $N^{\alpha\beta}$ and $M^{\alpha\beta}$. In fact there is an advantage in that both $\tilde{N}^{\alpha\beta}$ and $\tilde{M}^{\alpha\beta}$ are symmetric and thus there are fewer unknowns to deal with. This must simply mean that the equilibrium equations containing $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are slightly more general than is appropriate for a theory in which the displacements are restricted by the Kirchoff hypotheses. Moreover, in the present theory, there are the same number of stress quantities as there are strain quantities. If both a principle of minimum potential energy and a principle of minimum complementary energy (for an elastic shell) are to be possible of formulation in the theory, then it is ordinarily necessary that the constitutive relations be invertible. This is possible only if there are the same number of stress quantities as strain quantities.

The two strain tensors $E_{\alpha\beta}$ and $K_{\alpha\beta}$ (or equivalently the two tensors $G_{\alpha\beta}$ and $B_{\alpha\beta}$) furnish an adequate description of the deformation of the shell as shown by the following argument. In the first place the deformation of the shell is completely described in terms of the displacements of points on the middle surface provided the displacements throughout the thickness are restricted by the Kirchoff hypotheses. Secondly, from the theory of surfaces, a knowledge of $G_{\alpha\beta}$ and $B_{\alpha\beta}$ as functions of ξ^α and subject to the Gauss and Codazzi integrability conditions (which in the present case are equivalent to compatibility conditions) completely determines the deformed middle surface together with a coordinate system (the deformed ξ^α system)

except for a rigid body motion.

Modified Equilibrium Equations

Since new stress quantities have been introduced the equilibrium equations (23) to (26) are no longer appropriate for the theory being developed here. Appropriate equations can be derived from the expression (53) for internal virtual work.

$$
\begin{aligned}
\int_{\bar S} (\tilde N^{\alpha\beta}\delta E_{\alpha\beta}+\tilde M^{\alpha\beta}\delta K_{\alpha\beta})d\bar A &= \int_{\bar S} [\tilde N^{\alpha\beta}(\delta\bar U_{\beta/\alpha}+B_{\alpha\beta}\delta\bar W) \\
&+ \tilde M^{\alpha\beta}(-\delta\bar W_{/\alpha\beta}+B^{\gamma}_{\beta/\alpha}\delta\bar U_{\gamma}+2B^{\gamma}_{\beta}\delta\bar U_{\gamma/\alpha}+B^{\gamma}_{\alpha}B_{\gamma\beta}\delta\bar W)]d\bar A \\
&= \oint_{\bar C} [(\tilde N^{\alpha\beta}+B^{\alpha}_{\gamma}\tilde M^{\beta\gamma})\delta\bar U_{\alpha} + \tilde M^{\alpha\beta}_{/\alpha}\delta\bar W + \tilde M^{\alpha\beta}\delta\phi_{\alpha}]\bar n_{\beta}\, d\bar s \\
&- \int_{\bar S} [(\tilde N^{\alpha\beta}_{/\alpha}+2B^{\beta}_{\alpha}\tilde M^{\alpha\gamma}_{/\gamma}+B^{\beta}_{\gamma/\alpha}\tilde M^{\gamma\alpha})\delta\bar U_{\beta} \\
&+ (\tilde M^{\alpha\beta}_{/\alpha\beta}-B_{\alpha\beta}\tilde N^{\alpha\beta}-B^{\gamma}_{\alpha}B_{\gamma\beta}\tilde M^{\alpha\beta})\delta\bar W]d\bar A
\end{aligned}
\tag{54}
$$

The line integral around $\bar C$ is the external virtual work of the edge forces and moments. If a principle of virtual work is required to hold, and if the portion of the shell within $\bar C$ is in equilibrium, then the internal virtual work must equal the external virtual work for arbitrary virtual displacements. Thus the condition of equilibrium is that the last integral in (54) vanishes. This leads to the following equilibrium equations (surface forces have been omitted in this derivation for simplicity).

$$\tilde N^{\alpha\beta}_{/\alpha} + 2B^{\beta}_{\alpha}\tilde M^{\alpha\gamma}_{/\gamma} + B^{\beta}_{\gamma/\alpha}\tilde M^{\gamma\alpha} = 0 \tag{55}$$

$$\tilde M^{\alpha\beta}_{/\alpha\beta} - B_{\alpha\beta}\tilde N^{\alpha\beta} - B^{\gamma}_{\alpha}B_{\gamma\beta}\tilde M^{\alpha\beta} = 0 \tag{56}$$

Equations (55) and (56) are in fact identical to the equations (23) to (25) with Q^{α} eliminated. Equation (26) is accounted for by the symmetry of

$\tilde{N}^{\alpha\beta}$. If $\tilde{Q}^{\alpha}$ defined by

$$\tilde{Q}^{\alpha} = \tilde{M}^{\alpha\beta}_{/\beta} \tag{57}$$

is introduced as an approximation to Q^{α} , then the equilibrium equations, in an expanded form, may be written

$$\tilde{N}^{\alpha\beta}_{/\alpha} + B^{\beta}_{\alpha}\tilde{Q}^{\alpha} + (B^{\beta}_{\gamma}\tilde{M}^{\alpha\gamma})_{/\alpha} = 0 \tag{58}$$

$$\tilde{Q}^{\alpha}_{/\alpha} - B_{\alpha\beta}\tilde{N}^{\alpha\beta} - B^{\gamma}_{\alpha}B_{\beta\gamma}\tilde{M}^{\alpha\beta} = 0 \tag{59}$$

$$\tilde{M}^{\alpha\beta}_{/\alpha} - \tilde{Q}^{\beta} = 0 \tag{60}$$

The principle of virtual work reads

$$\begin{aligned}
&\oint_{\bar{C}} [(\tilde{N}^{\alpha\beta}+B^{\beta}_{\gamma}\tilde{M}^{\alpha\gamma})\delta\bar{U}_{\beta} + Q^{\alpha}\delta\bar{W} + \tilde{M}^{\alpha\beta}\delta\bar{\phi}_{\beta}]\bar{n}_{\alpha}\, d\bar{s} \\
&= \int_{\bar{S}} [\tilde{N}^{\alpha\beta}(\delta\bar{U}_{\beta/\alpha}+B_{\alpha\beta}\delta\bar{W}) + \tilde{Q}^{\alpha}(\delta\phi_{\alpha}+\delta\bar{W}_{/\alpha}-B^{\beta}_{\alpha}\delta\bar{U}_{\beta}) \\
&\quad + \tilde{M}^{\alpha\beta}(\delta\phi_{\beta/\alpha}+B^{\gamma}_{\beta}\delta\bar{U}_{\gamma/\alpha}+B^{\gamma}_{\alpha}B_{\beta\gamma}\delta\bar{W})]d\bar{A} \\
&= \int_{\bar{S}} (\tilde{N}^{\alpha\beta}\delta E_{\alpha\beta}+\tilde{Q}^{\alpha}\delta\gamma_{\alpha}+\tilde{M}^{\alpha\beta}\delta K_{\alpha\beta})d\bar{A}
\end{aligned} \tag{61}$$

It is something of a matter of personal preference, but it is this form of the principle of virtual work (with the $\tilde{Q}^{\alpha}$ and γ_{α} terms present) which will be used in the remainder of the paper.

SMALL STRAIN APPROXIMATION

Equilibrium Equations

In almost all practical applications of shell theory the middle surface strains are small whether or not the displacements are small, in which case the exact equations can be simplified somewhat. Since by definition the middle surface strain is

$$E_{\alpha\beta} = \frac{1}{2}(G_{\alpha\beta}-g_{\alpha\beta})$$

small middle surface strain means that the intrinsic geometry of the deformed middle surface is almost the same as that of the undeformed middle surface. Thus for sufficiently small middle surface strains (and exactly for inextensional bending) covariant differentiation with respect to $G_{\alpha\beta}$ may be replaced by covariant differentiation with respect to $g_{\alpha\beta}$. The equilibrium equations become

$$\begin{aligned} \tilde{N}^{\alpha\beta}_{,\alpha} + B^{\beta}_{\alpha}\tilde{Q}^{\alpha} + (B^{\beta}_{\gamma}\tilde{M}^{\alpha\gamma})_{,\alpha} + \bar{p}^{\beta} &= 0 \\ \tilde{Q}^{\alpha}_{,\alpha} - B_{\alpha\beta}\tilde{N}^{\alpha\beta} - B^{\gamma}_{\alpha}B_{\beta\gamma}\tilde{M}^{\alpha\beta} + \bar{p} &= 0 \qquad (62) \\ \tilde{M}^{\alpha\beta}_{,\alpha} - \tilde{Q}^{\beta} &= 0 \end{aligned}$$

<u>Strain-Displacement Relations</u>

A short investigation shows that the expression for $E_{\alpha\beta}$ given by (37) and (16) is not simplified at all by the assumption of small strains. The expression for $B_{\alpha\beta}$ given by (21) (and $K_{\alpha\beta}$ given by (50)) is slightly simplified because the expressions for $\cos\omega$ and ν^{α} in terms of $\lambda_{\alpha\beta}$ and μ_{α} (see equations (74) and (75) later in the paper) can be simplified by replacing $\sqrt{\frac{g}{G}}$ by unity. The expression for $K_{\alpha\beta}$ is obviously still quite complicated.

<u>Constitutive Relations</u>

Consistent constitutive relations for the linear small strain theory of thin elastic shells have been derived in references 11, 12 and 13. These derivations require only minor modifications in the case of finite displacements and small strains so they will not be reproduced here. For a thin shell of uniform thickness h composed of an isotropic hookean material, the constitutive relations are the same as in Love's first approximation, namely the <u>linear</u> relations

$$EhE_{\alpha\beta} = (1+\nu)g_{\alpha\gamma}g_{\beta\delta}\tilde{N}^{\gamma\delta} - \nu g_{\alpha\beta}g_{\gamma\delta}\tilde{N}^{\gamma\delta} \tag{63}$$

$$\frac{1}{12}\,Eh^3K_{\alpha\beta} = (1+\nu)g_{\alpha\gamma}g_{\beta\delta}\tilde{M}^{\gamma\delta} - \nu g_{\alpha\beta}g_{\gamma\delta}\tilde{M}^{\gamma\delta} \tag{64}$$

According to reference 11 these relations may be used even if the definition of $K_{\alpha\beta}$ in terms of displacements is altered by the addition of terms of the form $B^{\gamma}_{\alpha}E_{\gamma\beta}$. A similar argument to that in reference 11 shows that $\tilde{N}^{\alpha\beta}$ may be altered by addition of terms of the form $B^{\beta}_{\gamma}\tilde{M}^{\alpha\gamma}$. Note that in the case of small strain the indices on $\tilde{N}^{\alpha\beta}$ and $\tilde{M}^{\alpha\beta}$ may be raised and lowered with the metric $g_{\alpha\beta}$ instead of $G_{\alpha\beta}$ with negligible error.

If the material of the shell is not elastic and isotropic the relations (63) and (64) must be replaced by others appropriate for the material. However, the strain-displacement relations and the equilibrium equations given previously are unaffected by the material so long as transverse shear and normal strains can be neglected.

APPROXIMATION OF SMALL STRAINS AND MODERATELY SMALL ROTATIONS

The exact theory was considerably simplified by the assumption that the middle surface strains are small, but the equations are still very complicated. Considerable additional simplification can be achieved if the rotations are assumed small also. This simplification will be carried out in the following.

For infinitesimal displacements and rotations it is evident from (33) and (38) that the rotations are given by the formulas

$$\phi = \frac{1}{2}\epsilon^{\alpha\beta}U_{\beta,\alpha} \tag{65}$$

and

$$\phi_{\alpha} = -W_{,\alpha} + b^{\beta}_{\alpha}U_{\beta} = -\mu_{\alpha} \tag{66}$$

For small but finite rotations it is convenient to think of the expressions

in (65) and (66) as rotations (just as in the linear theory of shells). Purely for convenience, suppose that the coordinates ξ^α have the units of length so that ϕ, ϕ_α and $E_{\alpha\beta}$ are dimensionless. The following order of magnitude assumptions will lead to a theory one step beyond the linear theory in refinement.

$$\phi \quad \text{or} \quad \frac{1}{2}(U_{\alpha,\beta}-U_{\beta,\alpha}) = O(\varepsilon) \tag{67}$$

$$\phi_\alpha \quad \text{or} \quad \mu_\alpha = O(\varepsilon) \tag{68}$$

$$\frac{1}{2}(U_{\alpha,\beta}+U_{\beta,\alpha}) + b_{\alpha\beta}W = O(\varepsilon^2) \tag{69}$$

where ε is a number small compared to unity. Write $\lambda_{\alpha\beta}$ in the form

$$\begin{aligned}\lambda_{\alpha\beta} &= g_{\alpha\beta} + \frac{1}{2}(U_{\alpha,\beta}-U_{\beta,\alpha}) + \frac{1}{2}(U_{\alpha,\beta}+U_{\beta,\alpha}+2b_{\alpha\beta}W)\\ &= g_{\alpha\beta} - \varepsilon_{\alpha\beta}\phi + \frac{1}{2}(U_{\alpha,\beta}+U_{\beta,\alpha}+2b_{\alpha\beta}W)\end{aligned} \tag{70}$$

From (16) and (37) we find that $E_{\alpha\beta}$ is $O(\varepsilon^2)$ and is given approximately by the expression

$$E_{\alpha\beta} = \frac{1}{2}(U_{\alpha,\beta}+U_{\beta,\alpha}) + b_{\alpha\beta}W + \frac{1}{2}\mu_\alpha\mu_\beta + \frac{1}{2}g_{\alpha\beta}\phi^2 \tag{71}$$

The order of magnitude assumption (69) was made so that these terms would not dominate the expression for $E_{\alpha\beta}$; otherwise the linear theory would result.

In order to simplify the expression for $B_{\alpha\beta}$ expressions for ν_α and $\cos\omega$ are needed. An expression for $\bar{N}^i$ may be found by taking the cross product of the tangent vectors $y^i_{,\alpha}$.

$$\bar{\varepsilon}_{\alpha\beta}\bar{N}^k = \varepsilon_{ijk}y^i_{,\alpha}y^j_{,\beta} \tag{72}$$

Omitting the details, this leads to the following expression for $\bar{N}^i$ in terms of $\lambda_{\alpha\beta}$ and μ_α

$$N^i = \sqrt{\frac{g}{G}}\, \varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}(\tfrac{1}{2}\lambda_{\gamma\alpha}\lambda_{\delta\beta}N^i + \lambda_{\gamma\beta}\mu_\alpha x^i_{,\delta}) \tag{73}$$

From (13), (14) and (73)

$$\nu^\delta = \sqrt{\frac{g}{G}}\, \varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\lambda_{\gamma\beta}\mu_\alpha \tag{74}$$

$$\cos\omega = \frac{1}{2}\sqrt{\frac{g}{G}}\, \varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\lambda_{\gamma\alpha}\lambda_{\delta\beta} \tag{75}$$

From $G_{\alpha\beta} - g_{\alpha\beta} = O(\varepsilon^2)$ it follows that $\sqrt{\frac{g}{G}} = 1 + O(\varepsilon^2)$; then from (67) to (70) and (74) and (75) it follows that

$$\nu^\delta = -\mu^\delta + O(\varepsilon^2) \tag{76}$$

$$\cos\omega = 1 + O(\varepsilon^2) \tag{77}$$

From the foregoing and (21) the first approximation to $B_{\alpha\beta} - b_{\alpha\beta}$ is

$$\begin{aligned} T_{\alpha\beta} = B_{\alpha\beta} - b_{\alpha\beta} &\approx \frac{1}{2} b^\gamma_\beta(U_{\gamma,\alpha} - U_{\alpha,\gamma}) - \mu_{\alpha,\beta} \\ &= -W_{,\alpha\beta} + b^\gamma_\alpha U_{\gamma,\beta} + b^\gamma_{\alpha,\beta}U_\gamma + \frac{1}{2} b^\gamma_\beta(U_{\gamma,\alpha} - U_{\alpha,\gamma}) \end{aligned} \tag{78}$$

A difficulty here is that this expression is not symmetric in α and β . However, it can easily be shown that the antisymmetric part is negligible.

$$T_{\alpha\beta} - T_{\beta\alpha} = \frac{1}{2} b^\gamma_\alpha(U_{\gamma,\beta} + U_{\beta,\gamma}) - \frac{1}{2} b^\gamma_\beta(U_{\gamma,\alpha} + U_{\alpha,\gamma}) \tag{79}$$

Now $$\frac{1}{2}(U_{\alpha,\beta} + U_{\beta,\alpha}) + b_{\alpha\beta}W = O(\varepsilon^2)$$

so $$\frac{1}{2}(U_{\alpha,\beta} + U_{\beta,\alpha}) = -b_{\alpha\beta}W + O(\varepsilon^2) \tag{80}$$

and then $$T_{\alpha\beta} - T_{\beta\alpha} = (b^\gamma_\beta b_{\gamma\alpha} - b^\gamma_\alpha b_{\gamma\beta})W + \text{higher order terms} , \tag{81}$$

but the first term on the right hand side of (81) vanishes so $T_{\alpha\beta} - T_{\beta\alpha}$ is negligible (compared to $T_{\alpha\beta} + T_{\beta\alpha}$). The symmetric part of $T_{\alpha\beta}$ can be taken as the first approximation to the bending strain, giving

$$K_{\alpha\beta} = -\frac{1}{2}(\mu_{\alpha,\beta}+\mu_{\beta,\alpha}) + \frac{1}{2}(\varepsilon_{\alpha\gamma}b^{\gamma}_{\beta}+\varepsilon_{\beta\gamma}b^{\gamma}_{\alpha})\phi$$

or
$$K_{\alpha\beta} = \frac{1}{2}(\phi_{\alpha,\beta}+\phi_{\beta,\alpha}) + \frac{1}{4}b^{\gamma}_{\beta}(U_{\gamma,\alpha}-U_{\alpha,\gamma}) + \frac{1}{4}b^{\gamma}_{\alpha}(U_{\gamma,\beta}-U_{\beta,\gamma}) \tag{82}$$

Alternatively it can do no harm to include some higher order terms in $K_{\alpha\beta}$ if for some applications it simplifies the equations. The following approximation of $K_{\alpha\beta}$ may be used instead (see equation (21))

$$\begin{aligned} K_{\alpha\beta} &= b^{\gamma}_{\beta}\lambda_{\gamma\alpha} - \mu_{\alpha,\beta} - b_{\alpha\beta} \\ &= \phi_{\alpha,\beta} + b^{\gamma}_{\beta}U_{\gamma,\alpha} + b^{\gamma}_{\beta}b_{\gamma\alpha}W \\ &= -W_{,\alpha\beta} + b^{\gamma}_{\alpha,\beta}U_{\gamma} + b^{\gamma}_{\alpha}U_{\gamma,\beta} + b^{\gamma}_{\beta}U_{\gamma,\alpha} + b^{\gamma}_{\beta}b_{\gamma\alpha}W \end{aligned} \tag{83}$$

which is symmetric as it stands. Both alternative expressions are linear in the displacements. The expression (82) for the bending strain is the same as the one derived in references 11 and 14. The expression (83) does not seem to have appeared before in the literature. Either expression belongs to a consistent shell theory in the sense of Koiter (reference 11).

When transverse shear strains are neglected, the condition $\delta\gamma_{\alpha} = 0$ serves to relate $\delta\phi_{\alpha}$ to the displacements as in (39). By analogy the following expression should be used for γ_{α} in the present case

$$\gamma_{\alpha} = \phi_{\alpha} + \mu_{\alpha} = \phi_{\alpha} + W_{,\alpha} - b^{\gamma}_{\alpha}U_{\gamma} \tag{84}$$

Approximate Equilibrium Equations

Approximate equilibrium equations corresponding to the approximate strain-displacement equations may be found by the same method used to derive the equilibrium equations (55) and (56). Since the strains are small $d\bar{A}$ may be replaced by dA and we have

$$\int_S \Big\{ N^{\alpha\beta}[\delta U_{\alpha,\beta} + b_{\alpha\beta}\delta W + \mu_\alpha(\delta W_{,\beta} - b^\gamma_\beta \delta U_\gamma) + \frac{1}{2}\,\varepsilon_{\alpha\beta}\phi\varepsilon^{\gamma\delta}\delta U_{\delta,\gamma}]$$

$$+ M^{\alpha\beta}[\delta\phi_{\alpha,\beta} + \frac{1}{2}\,b^\gamma_\alpha(\delta U_{\gamma,\beta} - \delta U_{\beta,\gamma})] + Q^\alpha(\delta\phi_\alpha + \delta W_{,\alpha} - b^\gamma_\alpha \delta U_\gamma)\Big\}\,dA$$

$$= \oint_C - \int_S \Big\{ [N^{\alpha\beta}_{,\alpha} + b^\beta_\gamma \mu_\alpha N^{\alpha\gamma} + \frac{1}{2}\,\varepsilon^{\gamma\beta}(\phi N^\delta_\delta)_{,\gamma} + \frac{1}{2}(b^\beta_\alpha M^{\alpha\gamma})_{,\gamma} - \frac{1}{2}(b^\gamma_\alpha M^{\alpha\beta})_{,\gamma}$$

$$+ b^\beta_\alpha Q^\alpha]\delta U_\beta + [Q^\alpha_{,\beta} - b_{\alpha\beta}N^{\alpha\beta} + (\mu_\alpha N^{\alpha\beta})_{,\beta}]\delta W + [M^{\alpha\beta}_{,\beta} - Q^\alpha]\delta\phi_\alpha \Big\}\,dA \qquad (85)$$

By inspection the equilibrium equations are

$$N^{\alpha\beta}_{,\alpha} + b^\beta_\gamma \mu_\alpha N^{\alpha\gamma} + \frac{1}{2}\,\varepsilon^{\alpha\beta}(\phi N^\gamma_\gamma)_{,\alpha} + \frac{1}{2}(b^\beta_\alpha M^{\alpha\gamma})_{,\gamma} - \frac{1}{2}(b^\gamma_\alpha M^{\alpha\beta})_{,\gamma} + b^\beta_\alpha Q^\alpha + p^\beta = 0 \qquad (86)$$

$$Q^\alpha_{,\alpha} - b_{\alpha\beta}N^{\alpha\beta} + (\mu_\alpha N^{\alpha\beta})_{,\beta} + p = 0 \qquad (87)$$

$$M^{\alpha\beta}_{,\beta} - Q^\alpha = 0 \qquad (88)$$

where the load terms p^α and p have been supplied. These equations express equilibrium of forces and moments in directions parallel to the tangents and normal to the undeformed middle surface. In the left-hand side of (85) ϕ_α could be expressed in terms of displacements and the term $Q^\alpha \delta\gamma_\alpha$ could be omitted. The result for the equilibrium equations would be (86) and (87) with Q^α eliminated by means of (88). If the expression (83) is used for the bending strain, the first two equilibrium equations are slightly different and read as follows

$$N^{\alpha\beta}_{,\alpha} + b^\beta_\gamma \mu_\alpha N^{\alpha\gamma} + \frac{1}{2}\,\varepsilon^{\alpha\beta}(\phi N^\gamma_\gamma)_{,\alpha} + (b^\beta_\gamma M^{\alpha\gamma})_{,\alpha} + b^\beta_\alpha Q^\alpha + p^\beta = 0 \qquad (86)'$$

$$Q^\alpha_{,\alpha} - b_{\alpha\beta}N^{\alpha\beta} + (\mu_\alpha N^{\alpha\beta})_{,\beta} - b^\gamma_\beta b_{\gamma\alpha} M^{\alpha\beta} + p = 0 \qquad (87)'$$

Boundary Conditions

The Kirchoff boundary conditions may be obtained from the boundary integral in (85) which when written out reads

$$\oint_C \left\{ [N^{\alpha\beta} - \frac{1}{2}\,\varepsilon^{\alpha\beta}\phi N^{\gamma}_{\gamma} + \frac{1}{2}\,b^{\alpha}_{\gamma}M^{\gamma\beta} - \frac{1}{2}\,b^{\beta}_{\gamma}M^{\alpha\gamma}]\delta U_{\alpha} + (Q^{\beta} + \tilde{N}^{\alpha\beta}\mu_{\alpha})\delta W + \tilde{M}^{\alpha\beta}\delta\phi_{\alpha} \right\} n_{\beta}\, ds \tag{89}$$

Let t^{α} be the unit tangent to the curve C, then $n_{\beta} = \varepsilon_{\beta\gamma}t^{\gamma}$ is the unit normal to C in the surface S. Let

$$\phi_{\alpha} = \phi_s t_{\alpha} + \phi_n n_{\alpha} \tag{90}$$

where ϕ_s and ϕ_n are scalars. From (90)

$$\begin{aligned}\phi_s = \phi_{\alpha}t^{\alpha} &= (-W_{,\alpha} + b^{\gamma}_{\alpha}U_{\gamma})t^{\alpha} \\ &= -\frac{dW}{ds} + b^{\gamma}_{\alpha}U_{\gamma}t^{\alpha}\end{aligned} \tag{91}$$

Obviously ϕ_s is not independent of W and U_{α} on C. The last term in (89), namely,

$$\oint_C \tilde{M}^{\alpha\beta}\delta\phi_{\alpha}n_{\beta}\, ds = \oint_C \tilde{M}^{\alpha\beta}[-\frac{d\delta W}{ds} + b^{\gamma}_{\delta}\delta U_{\gamma}t^{\delta})t_{\alpha} + \delta\phi_n n_{\alpha}]n_{\beta}\, ds \tag{92}$$

becomes, upon integrating by parts,

$$\oint_C [\frac{d}{ds}(\tilde{M}^{\alpha\beta}t_{\alpha}n_{\beta})\delta W + \tilde{M}^{\gamma\beta}b^{\alpha}_{\delta}t^{\delta}t_{\gamma}n_{\beta}\delta U_{\alpha} + \tilde{M}^{\alpha\beta}n_{\alpha}n_{\beta}\delta\phi_n]ds \tag{93}$$

assuming C has a continuously turning tangent. Altogether (89) becomes

$$\begin{aligned}\oint_C \Big\{ &[N^{\alpha\beta} - \frac{1}{2}\,\varepsilon^{\alpha\beta}\phi N^{\gamma}_{\gamma} + \frac{1}{2}\,b^{\alpha}_{\gamma}\tilde{M}^{\gamma\beta} - \frac{1}{2}\,b^{\beta}_{\gamma}\tilde{M}^{\alpha\gamma} + b^{\alpha}_{\gamma}\tilde{M}^{\delta\beta}t^{\gamma}t_{\delta}]n_{\beta}\delta U_{\alpha} \\ &+ [Q^{\beta}n_{\beta} + \mu_{\alpha}\tilde{N}^{\alpha\beta}n_{\beta} + \frac{d}{ds}(\tilde{M}^{\alpha\beta}n_{\beta}t_{\alpha})]\delta W + \tilde{M}^{\alpha\beta}n_{\alpha}n_{\beta}\delta\phi_n \Big\} ds\end{aligned} \tag{94}$$

From this the boundary conditions on C may be read off. They are:

prescribe $[N^{\alpha\beta} - \frac{1}{2}\,\varepsilon^{\alpha\beta}\phi N^{\gamma}_{\gamma} + \frac{1}{2}\,b^{\alpha}_{\gamma}\tilde{M}^{\gamma\beta} - \frac{1}{2}\,b^{\beta}_{\gamma}\tilde{M}^{\alpha\gamma} + b^{\alpha}_{\gamma}\tilde{M}^{\delta\beta}t^{\gamma}t_{\delta}]n_{\beta}$ or U_{α} (95)

" $Q^{\beta}n_{\beta} + \mu_{\alpha}\tilde{N}^{\alpha\beta}n_{\beta} + \frac{d}{ds}(\tilde{M}^{\alpha\beta}n_{\beta}t_{\alpha})$ or W (96)

" $\tilde{M}^{\alpha\beta}n_{\alpha}n_{\beta}$ or ϕ_n (97)

For the alternative theory with (83) for the bending strain and

(86)', (87)' and (88) for equilibrium equations, the boundary conditions are:

$$\text{prescribe} \quad [N^{\alpha\beta} - \frac{1}{2}\varepsilon^{\alpha\beta}\phi M^{\gamma}_{\gamma} + b^{\alpha}_{\delta}(\delta^{\delta}_{\gamma}+t^{\delta}t_{\gamma})M^{\beta\gamma}]n_{\beta} \quad \text{or} \quad U_{\alpha} \tag{95'}$$

$$\text{"} \quad Q^{\alpha}n_{\alpha} + \mu_{\beta}N^{\alpha\beta}n_{\alpha} + \frac{d}{ds}(M^{\alpha\beta}t_{\alpha}n_{\beta}) \quad \text{or} \quad W \tag{96'}$$

$$\text{"} \quad M^{\alpha\beta}n_{\alpha}n_{\beta} \quad \text{or} \quad \phi_{n} \tag{97'}$$

FURTHER APPROXIMATIONS

Small Rotation About the Normal

If the rotation about the normal can be neglected compared to the other two rotations, then the equations can be simplified further. The importance of the rotation about the normal is not entirely established at the present time and no general condition under which it can be neglected is known. Several linear theories for thin shells have been constructed which differ from Love's first approximation only by terms in the bending strain proportional to the rotation about the normal. The differences between these theories and Love's are tabulated in reference 11 where the general validity of these theories is questioned. That the rotation about the normal can sometimes be neglected is evidenced by the fact that these theories lead to very nearly the same results as more accurate theories in some specific applications. See, for example, reference 15. On the other hand these theories lead to erroneous results in other applications. See references 16 and 17.

For those cases in which the approximation is valid the strains $E_{\alpha\beta}$ and $K_{\alpha\beta}$ given by equations (71) and (82) can be simplified to read

$$E_{\alpha\beta} = \frac{1}{2}(U_{\alpha,\beta}+U_{\beta,\alpha}) + b_{\alpha\beta}W + \frac{1}{2}\mu_{\alpha}\mu_{\beta} \tag{98}$$

$$K_{\alpha\beta} = \frac{1}{2}(\phi_{\alpha,\beta}+\phi_{\beta,\alpha}) \tag{99}$$

The corresponding approximate equilibrium equations (obtained via the virtual work principle) are

$$\tilde{N}^{\alpha\beta}_{,\beta} + b^{\alpha}_{\beta}\tilde{Q}^{\beta} - b^{\alpha}_{\beta}\mu_{\gamma}\tilde{N}^{\beta\gamma} + p^{\alpha} = 0 \tag{100}$$

$$\tilde{Q}^{\alpha}_{,\alpha} - b_{\alpha\beta}\tilde{N}^{\alpha\beta} + (\mu_{\alpha}\tilde{N}^{\alpha\beta})_{,\beta} + p = 0 \tag{101}$$

$$\tilde{M}^{\alpha\beta}_{,\beta} - \tilde{Q}^{\alpha} = 0 \tag{102}$$

and the boundary conditions are to prescribe:

$$[\tilde{N}^{\alpha\beta} + b^{\alpha}_{\delta}t^{\delta}t_{\gamma}\tilde{M}^{\gamma\beta}]n_{\beta} \quad \text{or} \quad U_{\alpha} \tag{103}$$

$$\tilde{Q}^{\beta}n_{\beta} + \mu_{\alpha}\tilde{N}^{\alpha\beta}n_{\beta} + \frac{d}{ds}(\tilde{M}^{\alpha\beta}t_{\alpha}n_{\beta}) \quad \text{or} \quad W \tag{104}$$

$$\tilde{M}^{\alpha\beta}n_{\alpha}n_{\beta} \quad \text{or} \quad \phi_{n} \tag{105}$$

When these equations are linearized they reduce, essentially, to those given in reference 10.

The Donnell-Mushtari-Vlasov Approximation

A further simplification of the above equations is possible under the assumptions discussed in reference 13. This consists in neglecting the term containing U_{α} in the expression for μ_{α} with the following results for strains,

$$E_{\alpha\beta} = \frac{1}{2}(U_{\alpha,\beta}+U_{\beta,\alpha}) + b_{\alpha\beta}W + \frac{1}{2}W_{,\alpha}W_{,\beta} \tag{106}$$

$$K_{\alpha\beta} = \frac{1}{2}(\phi_{\alpha,\beta}+\phi_{\beta,\alpha}) = -W_{,\alpha\beta} \tag{107}$$

$$\gamma_{\alpha} = \phi_{\alpha} + W_{,\alpha} \tag{108}$$

for equilibrium equations,

$$\tilde{N}^{\alpha\beta}_{,\beta} + p^{\alpha} = 0 \tag{109}$$

$$\tilde{Q}^{\alpha}_{,\alpha} - b_{\alpha\beta}\tilde{N}^{\alpha\beta} + (W_{,\alpha}\tilde{N}^{\alpha\beta})_{,\beta} + p = 0 \tag{110}$$

$$\tilde{M}^{\alpha\beta}_{,\beta} - \tilde{Q}^{\alpha} = 0 \tag{111}$$

and for boundary conditions prescribe

$$\tilde{N}^{\alpha\beta} n_{\beta} \quad \text{or} \quad U_{\alpha} \tag{112}$$

$$(\tilde{Q}^{\beta} + W_{,\alpha}\tilde{N}^{\alpha\beta})n_{\beta} + \frac{d}{ds}(\tilde{M}^{\alpha\beta} t_{\alpha} n_{\beta}) \quad \text{or} \quad W \tag{113}$$

$$\tilde{M}^{\alpha\beta} n_{\alpha} n_{\beta} \quad \text{or} \quad \phi_{n} \tag{114}$$

Marguerre's Shallow Shell Equations

If applied to a shallow shell the preceding equations can be further simplified because of the geometry. Suppose that the shell is nearly flat and parallel to the $x^3 = z = 0$ plane, and that the squares of the slopes of the shell with respect to the $z = 0$ plane may be neglected. Then, approximately:

$$b_{\alpha\beta} = -z_{,\alpha\beta} \tag{115}$$

Since the displacements U_{α} are considered small compared to W, the horizontal displacements $\bar{U}_{\alpha}$ and the vertical displacements $\bar{W}$ are given approximately in terms of U_{α} and W by

$$\begin{aligned} U_{\alpha} &\approx \bar{U}_{\alpha} + z_{,\alpha}\bar{W} \\ W &\approx \bar{W} \end{aligned} \tag{116}$$

In terms of $\bar{U}_{\alpha}$ and W the membrane strain $E_{\alpha\beta}$ (equation (106)) becomes:

$$E_{\alpha\beta} = \frac{1}{2}(\bar{U}_{\alpha,\beta} + \bar{U}_{\beta,\alpha} + z_{,\alpha}W_{,\beta} + z_{,\beta}W_{,\alpha} + W_{,\alpha}W_{,\beta}) \tag{117}$$

The strain $K_{\alpha\beta}$ and γ_{α} are as before (in equations (107) and (108)). The conditions of equilibrium in the horizontal and vertical directions are

$$\tilde{N}^{\alpha\beta}_{,\beta} + \bar{p}^{\alpha} = 0 \tag{118}$$

$$\tilde{Q}^{\alpha}_{,\alpha} + [(z_{,\alpha} + W_{,\alpha})\tilde{N}^{\alpha\beta}]_{,\beta} + \bar{p} = 0 \tag{119}$$

$$M^{\alpha\beta}_{,\beta} - Q^{\alpha} = 0 \tag{120}$$

where $\bar{p}^{\alpha}$ is the horizontal load intensity and $\bar{p}$ is the vertical load intensity. The boundary conditions are to prescribe

$$N^{\alpha\beta}n_{\beta} \quad \text{or} \quad U_{\alpha} \tag{121}$$

$$[Q^{\beta} + (z_{,\alpha}+W_{,\alpha})N^{\alpha\beta}]n_{\beta} + \frac{d}{ds}(M^{\alpha\beta}t_{\alpha}n_{\beta}) \quad \text{or} \quad W \tag{122}$$

$$M^{\alpha\beta}n_{\alpha}n_{\beta} \quad \text{or} \quad \phi_{n} \tag{123}$$

These are Marguerre's shallow shell equations in tensor form (reference 1).

CONCLUDING REMARKS

Several nonlinear theories for thin shells have been derived in increasing stages of approximation. The linearization of these equations, which is more or less obvious, has been omitted but in most cases the resulting linear equations are essentially the same as shell equations already given in the literature. In all cases the theories are first approximation theories in the sense that transverse shear and normal strains are neglected.

In each of the theories derived in this paper the equilibrium equations and strain-displacement relations are related by a principle of virtual work and hence the usual variational principles may be formulated and proved. These derivations are also more or less obvious and have been omitted.

The additional manipulations necessary to apply the equations to stability problems has not been given either, but the process is well known and, of course, different manipulations may be required in different special cases.

SHELL EQUATIONS IN ORDINARY NOTATION

In the ordinary notation with lines of curvature for coordinates (as used in references 9 and 14) the expressions (71) for middle surface strains and (82) for bending strains are, respectively

$$
\begin{aligned}
\epsilon_{11} &= \frac{1}{\alpha_1}\frac{\partial U_1}{\partial \xi_1} + \frac{1}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} U_2 + \frac{W}{R_1} + \frac{1}{2}\mu_1^2 + \underline{\frac{1}{2}\phi^2} \\
\epsilon_{22} &= \frac{1}{\alpha_2}\frac{\partial U_2}{\partial \xi_2} + \frac{1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1} U_1 + \frac{W}{R_2} + \frac{1}{2}\mu_2^2 + \underline{\frac{1}{2}\phi^2} \qquad \text{(A-1)} \\
\epsilon_{12} &= \frac{1}{2}\left(\frac{1}{\alpha_1}\frac{\partial U_2}{\partial \xi_1} + \frac{1}{\alpha_2}\frac{\partial U_1}{\partial \xi_2} - \frac{1}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} U_1 - \frac{1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1} U_2 + \mu_1\mu_2\right)
\end{aligned}
$$

$$
\begin{aligned}
\kappa_{11} &= \frac{1}{\alpha_1}\frac{\partial \phi_1}{\partial \xi_1} + \frac{1}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2}\phi_2 \\
\kappa_{22} &= \frac{1}{\alpha_2}\frac{\partial \phi_2}{\partial \xi_2} + \frac{1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1}\phi_1 \qquad \text{(A-2)} \\
\kappa_{12} &= \frac{1}{2}\left[\frac{1}{\alpha_1}\frac{\partial \phi_2}{\partial \xi_1} + \frac{1}{\alpha_2}\frac{\partial \phi_1}{\partial \xi_2} - \frac{1}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2}\phi_1 - \frac{1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1}\phi_2 + \underline{\frac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\phi}\right]
\end{aligned}
$$

The transverse shear strains are given by

$$
\begin{aligned}
\gamma_1 &= \frac{1}{\alpha_1}\frac{\partial W}{\partial \xi_1} \underline{- \frac{U_1}{R_1}} + \phi_1 \\
& \qquad\qquad\qquad\qquad \text{(A-3)} \\
\gamma_2 &= \frac{1}{\alpha_2}\frac{\partial W}{\partial \xi_2} \underline{- \frac{U_2}{R_2}} + \phi_2
\end{aligned}
$$

The rotations are given by

$$
\begin{aligned}
\phi_1 &= -\mu_1 = -\frac{1}{\alpha_1}\frac{\partial W}{\partial \xi_1} \underline{+ \frac{U_1}{R_1}} \\
\phi_2 &= -\mu_2 = -\frac{1}{\alpha_2}\frac{\partial W}{\partial \xi_2} \underline{+ \frac{U_2}{R_2}} \qquad \text{(A-4)} \\
\phi &= \frac{1}{2\alpha_1\alpha_2}\left(\frac{\partial \alpha_2 U_2}{\partial \xi_1} - \frac{\partial \alpha_1 U_1}{\partial \xi_2}\right)
\end{aligned}
$$

The equilibrium equations (86), (87), and (88) now read

$$\frac{\partial \alpha_2 N_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{12}}{\partial \xi_2} + \frac{\partial \alpha_1}{\partial \xi_2} N_{12} - \frac{\partial \alpha_2}{\partial \xi_1} N_{22} + \frac{\alpha_1 \alpha_2}{R_1} Q_1$$

$$+ \frac{\alpha_1}{2} \frac{\partial}{\partial \xi_2} \left[\left(\frac{1}{R_1} - \frac{1}{R_2} \right) M_{12} \right] + \frac{\alpha_1 \alpha_2}{R_1} (\mu_1 N_{11} + \mu_2 N_{12})$$

$$- \frac{\alpha_1}{2} \frac{\partial}{\partial \xi_2} \left[(N_{11} + N_{22}) \phi \right] + \alpha_1 \alpha_2 p_1 = 0 \tag{A-5}$$

$$\frac{\partial \alpha_2 N_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{22}}{\partial \xi_2} + \frac{\partial \alpha_2}{\partial \xi_1} N_{12} - \frac{\partial \alpha_1}{\partial \xi_2} N_{11} + \frac{\alpha_1 \alpha_2}{R_2} Q_2$$

$$+ \frac{\alpha_2}{2} \frac{\partial}{\partial \xi_1} \left[\left(\frac{1}{R_2} - \frac{1}{R_1} \right) M_{12} \right] + \frac{\alpha_1 \alpha_2}{R_2} (\mu_1 N_{12} + \mu_2 N_{22})$$

$$+ \frac{\alpha_2}{2} \frac{\partial}{\partial \xi_1} \left[(N_{11} + N_{22}) \phi \right] + \alpha_1 \alpha_2 p_2 = 0 \tag{A-6}$$

$$\frac{\partial \alpha_2 Q_1}{\partial \xi_1} + \frac{\partial \alpha_1 Q_2}{\partial \xi_2} - \alpha_1 \alpha_2 \left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} \right) + \frac{\partial}{\partial \xi_1} (\alpha_2 \mu_1 N_{11} + \alpha_2 \mu_2 N_{12})$$

$$+ \frac{\partial}{\partial \xi_2} (\alpha_1 \mu_1 N_{12} + \alpha_1 \mu_2 N_{22}) + \alpha_1 \alpha_2 p_n = 0 \tag{A-7}$$

$$\frac{\partial \alpha_2 M_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{12}}{\partial \xi_2} + \frac{\partial \alpha_1}{\partial \xi_2} M_{12} - \frac{\partial \alpha_2}{\partial \xi_1} M_{22} - \alpha_1 \alpha_2 Q_1 = 0 \tag{A-8}$$

$$\frac{\partial \alpha_2 M_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{22}}{\partial \xi_2} + \frac{\partial \alpha_2}{\partial \xi_1} M_{12} - \frac{\partial \alpha_1}{\partial \xi_2} M_{11} - \alpha_1 \alpha_2 Q_2 = 0 \tag{A-9}$$

The boundary conditions on an edge ξ_1 = constant are to prescribe

$$N_{11} \quad \text{or} \quad U_1$$

$$N_{12} + \underline{\left(\frac{3}{2R_2} - \frac{1}{2R_1}\right)}M_{12} + \underline{\frac{1}{2}(N_{11}+N_{22})\phi} \quad \text{or} \quad U_2$$

$$Q_1 + \frac{1}{\alpha_2}\frac{\partial M_{12}}{\partial \xi_2} + \mu_1 N_{11} + \mu_2 N_{12} \quad \text{or} \quad W \tag{A-10}$$

$$M_{11} \quad \text{or} \quad \phi_1$$

The boundary conditions on an edge ξ_2 = constant are the same as these with the subscripts 1 and 2 interchanged except for the second one in which the sign of the term involving ϕ should be changed.

The terms in the preceding equations which drop out when rotations around the normal are neglected have a solid underline. The terms which drop out in the Donnell-Mushtari-Vlasov approximation are those with either a solid underline or a dashed underline.

The bending strains of the alternative theory (equation (83)) in the present notation are

$$\kappa_{11} = \frac{1}{\alpha_1}\frac{\partial \phi_1}{\partial \xi_1} + \frac{1}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2}\phi_2 + \frac{1}{R_1}\left(\frac{1}{\alpha_1}\frac{\partial U_1}{\partial \xi_1} + \frac{1}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2}U_2\right) + \frac{W}{R_1^2}$$

$$\kappa_{22} = \frac{1}{\alpha_2}\frac{\partial \phi_2}{\partial \xi_2} + \frac{1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1}\phi_1 + \frac{1}{R_2}\left(\frac{1}{\alpha_2}\frac{\partial U_2}{\partial \xi_2} + \frac{1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1}U_1\right) + \frac{W}{R_2^2} \tag{A-11}$$

$$\kappa_{12} = \frac{1}{2}\left[\frac{1}{\alpha_2}\frac{\partial \phi_1}{\partial \xi_2} + \frac{1}{\alpha_1}\frac{\partial \phi_2}{\partial \xi_1} - \frac{1}{\alpha_1\alpha_2}\left(\frac{\partial \alpha_1}{\partial \xi_2}\phi_1 + \frac{\partial \alpha_2}{\partial \xi_1}\phi_2\right)\right.$$

$$\left. + \frac{1}{R_1}\left(\frac{1}{\alpha_2}\frac{\partial U_1}{\partial \xi_2} - \frac{1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1}U_2\right) + \frac{1}{R_2}\left(\frac{1}{\alpha_1}\frac{\partial U_2}{\partial \xi_1} - \frac{1}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2}U_1\right)\right]$$

The membrane strains are the same as those given by (A-1). The force equilibrium equations are

$$\frac{\partial\alpha_2\tilde{N}_{11}}{\partial\xi_1} + \frac{\partial\alpha_1\tilde{N}_{12}}{\partial\xi_2} + \frac{\partial\alpha_1}{\partial\xi_2}\tilde{N}_{12} - \frac{\partial\alpha_2}{\partial\xi_1}\tilde{N}_{22} + \frac{\alpha_1\alpha_2}{R_1}\tilde{Q}_1 + \frac{\partial}{\partial\xi_1}(\frac{\alpha_2}{R_1}\tilde{M}_{11})$$

$$- \frac{1}{R_2}\frac{\partial\alpha_2}{\partial\xi_1}\tilde{M}_{22} + \frac{1}{R_2}\frac{\partial\alpha_1}{\partial\xi_2}\tilde{M}_{12} + \frac{\partial}{\partial\xi_2}(\frac{\alpha_1}{R_1}\tilde{M}_{12})$$

$$+ \frac{\alpha_1\alpha_2}{R_1}(\mu_1\tilde{N}_{11}+\mu_2\tilde{N}_{12}) - \frac{\alpha_1}{2}\frac{\partial}{\partial\xi_2}[(\tilde{N}_{11}+\tilde{N}_{22})\phi] + \alpha_1\alpha_2 p_1 = 0 \qquad \text{(A-12)}$$

$$\frac{\partial\alpha_2\tilde{N}_{12}}{\partial\xi_1} + \frac{\partial\alpha_1\tilde{N}_{22}}{\partial\xi_2} + \frac{\partial\alpha_2}{\partial\xi_1}\tilde{N}_{12} - \frac{\partial\alpha_1}{\partial\xi_2}\tilde{N}_{11} + \frac{\alpha_1\alpha_2}{R_2}\tilde{Q}_2 + \frac{\partial}{\partial\xi_2}(\frac{\alpha_1}{R_2}\tilde{M}_{22})$$

$$- \frac{1}{R_1}\frac{\partial\alpha_1}{\partial\xi_2}\tilde{M}_{11} + \frac{\partial}{\partial\xi_1}(\frac{\alpha_2}{R_2}\tilde{M}_{12}) + \frac{1}{R_1}\frac{\partial\alpha_2}{\partial\xi_1}\tilde{M}_{12}$$

$$+ \frac{\alpha_1\alpha_2}{R_2}(\mu_1\tilde{N}_{12}+\mu_2\tilde{N}_{22}) + \frac{\alpha_2}{2}\frac{\partial}{\partial\xi_1}[(\tilde{N}_{11}+\tilde{N}_{22})\phi] + \alpha_1\alpha_2 p_2 = 0 \qquad \text{(A-13)}$$

$$\frac{\partial\alpha_2\tilde{Q}_1}{\partial\xi_1} + \frac{\partial\alpha_1\tilde{Q}_2}{\partial\xi_2} - \alpha_1\alpha_2(\frac{\tilde{N}_{11}}{R_1} + \frac{\tilde{N}_{22}}{R_2}) - \alpha_1\alpha_2(\frac{\tilde{M}_{11}}{R_1^2} + \frac{\tilde{M}_{22}}{R_2^2})$$

$$+ \frac{\partial}{\partial\xi_1}(\alpha_2\mu_1\tilde{N}_{11}+\alpha_2\mu_2\tilde{N}_{12}) + \frac{\partial}{\partial\xi_2}(\alpha_1\mu_1\tilde{N}_{12}+\alpha_1\mu_2\tilde{N}_{22}) + \alpha_1\alpha_2 p_n = 0 \quad \text{(A-14)}$$

The moment equilibrium equations are the same as (A-8) and (A-9).

On an edge ξ_1 = constant the boundary conditions are to prescribe

$$\begin{aligned}
&\tilde{N}_{11} + \frac{\tilde{M}_{11}}{R_1} \quad \text{or} \quad U_1 \\
&\tilde{N}_{12} + \frac{2}{R_2}\tilde{M}_{12} + \frac{1}{2}(\tilde{N}_{11}+\tilde{N}_{22})\phi \quad \text{or} \quad U_2 \\
&\tilde{Q}_1 + \frac{1}{\alpha_2}\frac{\partial\tilde{M}_{12}}{\partial\xi_2} + \mu_1\tilde{N}_{11} + \mu_2\tilde{N}_{12} \quad \text{or} \quad W \\
&\tilde{M}_{11} \quad \text{or} \quad \phi_1
\end{aligned} \qquad \text{(A-15)}$$

On an edge ξ_2 = constant the boundary conditions are the same as these with the subscripts 1 and 2 interchanged except in the second line the term with ϕ should be changed in sign.

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